Revisiting the Poincaré Sphere as a Representation of Polarization State

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Abstract – **Graphical representations of the polarization state of an antenna or an electromagnetic wave propagating through space are useful tools to supplement rigorous mathematical analyses. One such example, the polarization ellipse, is frequently used in combination with the mathematical development of polarization theory.**

The Poincaré sphere is another graphical representation but is much less widely used. Since each possible polarization state appears as a point on the surface of the sphere, it has limited value in representing a single polarization state. However, it can be quite useful for visualizing the relationships between multiple polarization states.

In this paper, we show a different way of presenting the Poincaré sphere using a Mercator projection and elliptical parameters. We also describe a tool that implements this technique and provides a real-time display of polarization state as a function of frequency.

Index Terms— **Polarization, Poincaré sphere, Dual-pol probe, Calibration, Antenna measurements**

I. Introduction

The concepts presented in this paper grew from prior work done to correct polarization distortion in a compact range feed [1], where a need arose to graphically display and compare polarization states. The display format and approach presented here evolved during the process of developing the Polarization Analyzer tool described in [2].

The Poincaré sphere is difficult to represent robustly in three dimensions because data points may appear on the back side of the sphere, depending on the perspective of the rendering. In this paper, we use a Mercator projection to create a two-dimensional plot so that the location of points on the sphere are more readily perceived without ambiguity. The latitude and longitude axes are shown to be equivalent to the familiar elliptical polarization parameters of axial ratio, tilt angle, and sense. Since each polarization state is represented by a single point on the sphere, this representation can be used to show how the polarization state changes as a function of some independent variable, such as frequency.

The remainder of the paper is organized as follows. In Section II, we give a brief introduction to polarization theory. We then explore several mathematical representations of polarization state including: Jones vectors and the complex polarization ratio (Section III), Stokes parameters (Section IV), and elliptical parameters (Section VI). The Stokes parameters are used as a basis to describe the Poincaré sphere in Section V and the final form of the Poincaré sphere with elliptical parameters is given in Section VI.

Using this alternate Poincaré sphere representation, the display of polarization state across multiple frequencies has been implemented in the Polarization Analyzer tool. The result is described in Section VII, followed by a summary in Section VIII.

II. Polarization Theory

The electric field of a single-frequency propagating plane wave may be written as [3]

$$\vec{E}(\vec{x},t) = (E_1\hat{e}_1 + E_2\hat{e}_2)e^{j(\omega t - \vec{k}\cdot\vec{x})} \tag{1}$$

Here, E_1 and E_2 are complex projections of the field onto two polarization bases at some point in time and space where $\omega t = \vec{k}\cdot\vec{x}$. In general, these bases can be any set of orthogonal pairs, but for convenience, we will let E_1 indicate horizontal polarization and E_2 vertical. And again, for convenience, we will let the direction of propagation be along the z-axis in the positive direction. Given these assumptions ($\hat{e}_1 = \hat{e}_x$, $\hat{e}_2 = \hat{e}_y$, $\vec{x} = z\hat{e}_z$, and $\vec{k} = k\hat{e}_z$), we can rewrite the E-field as

$$\vec{E}(z,t) = \left(E_1\hat{e}_x + E_2\hat{e}_y\right)e^{j(\omega t - kz)} \tag{2}$$

We can write the projected fields generically as

$$E_1 = a_1 e^{j\phi_1} \tag{3}$$

$$E_2 = a_2 e^{j\phi_2} \tag{4}$$

We can then write the field components of the wave as follows:

$$\begin{aligned} E_x(z,t) &= \mathcal{R}e\{\vec{E}(z,t)\cdot\hat{e}_x\} = \mathcal{R}e\{E_1 e^{j(\omega t - kz)}\} \\ &= a_1\cos(\omega t - kz + \phi_1) \end{aligned} \tag{5}$$

$$\begin{aligned} E_y(z,t) &= \mathcal{R}e\{\vec{E}(z,t)\cdot\hat{e}_y\} = \mathcal{R}e\{E_2 e^{j(\omega t - kz)}\} \\ &= a_2\cos(\omega t - kz + \phi_2) \end{aligned} \tag{6}$$

At a fixed point in space along the z-axis, z_0, if we let t vary over an interval of time equal to λ/c, the field will trace an ellipse as it propagates through the plane defined by $z = z_0$. Here λ is the wavelength of the single-frequency wave and c is the speed of the propagating wave. Thus, we let time progress until one wavelength has passed through the plane defined by $z = z_0$.

Figure 1 below shows an example of such an ellipse, traced by a wave whose polarization is defined by $E_1 = 1$ and $E_2 = 0.6e^{j(0.4\pi)}$.

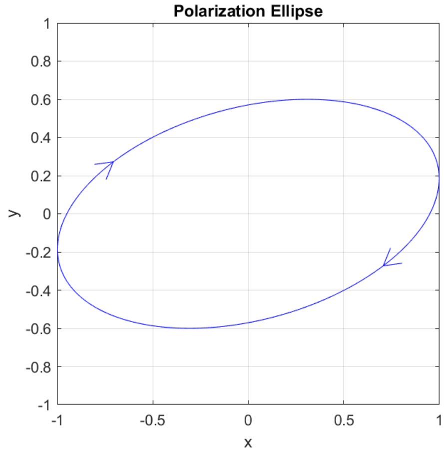

Figure 1. Polarization ellipse associated with $E_1 = 1$ *and* $E_2 = 0.6e^{j(0.4\pi)}$

Figure 2 shows a 3-D plot of the wave at a single point in time. The z-axis is shown in units of wavelengths, and a single cross-section at $z = \lambda$ indicates a slice of space where a polarization ellipse is traced. In this case, the field at $z = \lambda$ appears to rotate clockwise from the current plot perspective. The wave propagates in the positive z direction, as indicated by the large blue arrow.

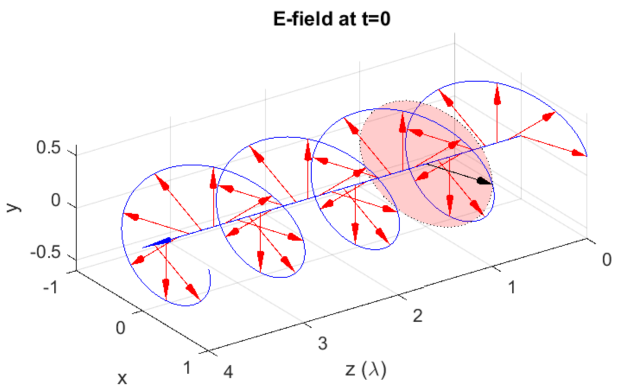

Figure 2. 3-dimensional wave representation for $E_1 = 1$ *and* $E_2 = 0.6e^{j(0.4\pi)}$ *at time t=0*

Note that the field vectors, some of which are indicated on the figure with red arrows, are the vectors computed from the field components in the equations above at various values of z for $t = 0$. The field vector passing through the plane $z = \lambda$ is colored black. As t progresses, these field vectors propagate along the axis of propagation.

III. Jones Vector and Complex Polarization Ratio

The vector $\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$ is called a Jones vector and contains all of the information necessary to represent the polarization state of the electromagnetic wave. Another representation that completely describes the polarization state is the ratio of these two values, sometimes called the complex polarization ratio, given by

$$\rho = \frac{E_2}{E_1} = \frac{a_2}{a_1} e^{j(\phi_2 - \phi_1)} \tag{7}$$

Other representations of polarization state include Stokes parameters and elliptical parameters of axial ratio, tilt, and sense.

IV. Stokes Parameters

Stokes parameters are a set of four parameters that originated from the study of optics in the mid-19th century [4]. The first parameter is a measure of the intensity of the wave. The other three are a set of coefficients defining how the polarization state of a wave relates to three different sets of orthogonal polarizations. They are defined as follows.

As stated above, the first parameter is the energy, or intensity, of the wave. It is written as

$$S_0 = |E_H|^2 + |E_V|^2 \tag{8}$$

The second parameter is a measure of the energy of the wave projected onto horizontal and vertical polarizations. By taking the difference in power between these projections, we obtain a measure of how well the wave couples to each. This parameter is written as

$$S_1 = |E_H|^2 - |E_V|^2 \tag{9}$$

The third parameter is defined similarly, but instead of horizontal and vertical, it is the difference in power between the projection onto a Slant+45 polarization and a Slant-45 polarization. It is written as

$$\begin{aligned} S_2 &= |E_{+45}|^2 - |E_{-45}|^2 \\ &= \left|\frac{E_H + E_V}{\sqrt{2}}\right|^2 - \left|\frac{E_H - E_V}{\sqrt{2}}\right|^2 \\ &= \frac{1}{2}[(E_H + E_V)(E_H^* + E_V^*) - (E_H - E_V)(E_H^* - E_V^*)] \\ &= \frac{1}{2}[[|E_H|^2 + E_H^* E_V + E_H E_V^* + |E_V|^2] \\ &\qquad - [|E_H|^2 - E_H^* E_V - E_H E_V^* + |E_V|^2]] \\ &= E_H^* E_V + E_H E_V^* \\ &= 2\mathcal{R}e\{E_H^* E_V\} \\ &= 2a_1 a_2 \cos(\phi_2 - \phi_1) \end{aligned}$$

The fourth parameter is defined similarly, but it is the difference in power between the projection onto a left-hand circular polarization and a right-hand circular polarization. It is written as

$$\begin{aligned} S_3 &= |E_{LHCP}|^2 - |E_{RHCP}|^2 \\ &= \left|\frac{E_H - jE_V}{\sqrt{2}}\right|^2 - \left|\frac{E_H + jE_V}{\sqrt{2}}\right|^2 \\ &= \frac{1}{2}[(E_H - jE_V)(E_H^* + jE_V^*) - (E_H + jE_V)(E_H^* - jE_V^*)] \\ &= \frac{1}{2}[[|E_H|^2 - jE_H^* E_V + jE_H E_V^* + |E_V|^2] \\ &\qquad - [|E_H|^2 + jE_H^* E_V - jE_H E_V^* + |E_V|^2]] \\ &= -j(E_H^* E_V - E_H E_V^*) \\ &= 2\mathcal{I}m\{E_H^* E_V\} \\ &= 2a_1 a_2 \sin(\phi_2 - \phi_1) \end{aligned}$$

It is often convenient to normalize these four parameters by s_0 so we only have to deal with three:

$$s_1 = \frac{S_1}{S_0}, s_2 = \frac{S_2}{S_0}, s_3 = \frac{S_3}{S_0} \tag{10}$$

Note that

$$\begin{aligned} &S_1^2 + S_2^2 + S_3^2 \\ &= (|E_H|^2 - |E_V|^2)^2 + (|E_{+45}|^2 - |E_{-45}|^2)^2 \\ &\qquad + (|E_{LHCP}|^2 - |E_{RHCP}|^2)^2 \end{aligned}$$

$$
\begin{aligned}
&= (|E_H|^2 - |E_V|^2)^2 + (E_H^* E_V + E_H E_V^*)^2 \\
&\qquad + \left(-j(E_H^* E_V - E_H E_V^*)\right)^2 \\
&= (|E_H|^4 - 2|E_H|^2|E_V|^2 + |E_V|^4) \\
&\qquad + \left(E_H^{*\,2} E_V^2 + 2|E_H E_V|^2 + E_H^2 E_V^{*\,2}\right) \\
&\qquad - \left(E_H^{*\,2} E_V^2 - 2|E_H E_V|^2 + E_H^2 E_V^{*\,2}\right) \\
&= |E_H|^4 - 2|E_H|^2|E_V|^2 + |E_V|^4 + 4|E_H E_V|^2 \\
&= |E_H|^4 + 2|E_H|^2|E_V|^2 + |E_V|^4 \\
&= (|E_H|^2 + |E_V|^2)^2 = S_0^2
\end{aligned}
$$

Since $S_0^2 = S_1^2 + S_2^2 + S_3^2$, we can say of the normalized Stokes parameters that

$$s_1^2 + s_2^2 + s_3^2 = 1 \tag{11}$$

V. The Poincaré Sphere

The three normalized Stokes parameters may be thought of as coefficients along three orthogonal axes. Let s_1 be the coefficient along the x-axis, s_2 the coefficient along the y-axis, and s_3 along the z-axis. Given the relationship in (11), we see that these three coefficients become the Cartesian coordinates of a point on the unit sphere. We call this sphere the Poincaré sphere [5].

The Poincaré sphere's representation can be quite useful for understanding how polarization varies with respect to some independent variable. For example, by plotting polarization state on the sphere, we can quickly understand how polarization varies with frequency. Or with angular position (i.e. how does polarization change across the pattern of an antenna?). Or any other variable we choose.

Since the three Cartesian coordinates are defined as projections onto certain standard bases, we can develop some intuition for the polarization state of a wave represented by a point on the sphere. The table below summarizes the six polarization bases used to generate the Stokes parameters and their positions in the Cartesian space that includes the Poincaré sphere.

Table 1. Normalized Stokes parameter values for the six standard polarization bases

	s_1	s_2	s_3
Horizontal	1	0	0
Vertical	-1	0	0
Slant+45	0	1	0
Slant-45	0	-1	0
LHCP	0	0	1
RHCP	0	0	-1

Based on the table and considering how the Stokes parameters are generated, we can surmise several things about the Poincaré sphere:

1. Horizontal polarization is along the positive x-axis, vertical along the negative x-axis
2. Slant+45 is along the positive y-axis, Slant-45 along the negative y-axis
3. LHCP is along the positive z-axis, RHCP along the negative z-axis
4. Linear polarizations lie on the equator of the sphere, in the xy-plane
5. Ellipticity increases as we move away from the equator and toward the poles, where it becomes circular
6. Orthogonal polarizations lie on opposite sides of the sphere

An example plot of a Poincaré sphere is shown in Figure 3. In the plot, the horizontal axis is the y-axis and the vertical axis is the z-axis. The positive x-axis is coming out of the page. On the sphere, we see polarization states of a single antenna's H-pol boresight response for several different frequencies. For reference, note that longitudinal lines are spaced every 20 degrees and latitude lines every 10 degrees.

We can see that the polarizations are largely linear (close to the equator) but have a small amount of ellipticity. As frequency varies, the tilt of the polarization changes, moving more or less along the equator, but deviating from pure horizontal, which is in the exact center of the figure, represented by a black circle.

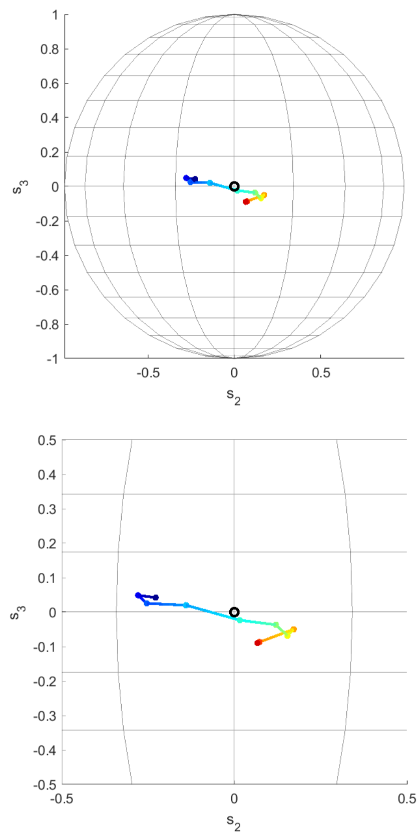

Figure 3. Poincaré sphere plot showing polarization state vs frequency for a measured antenna (top: full-scale, bottom: zoomed in)

In this figure, we also show the polarization states in a progressive color scheme where the lowest frequency is given in red and the highest frequency in purple, mimicking the frequency-dependent progression of colors in the rainbow associated with the optical frequencies. This allows us to easily identify the evolution of polarization state versus frequency. If polarization is plotted versus another variable, such as aspect angle, the colors would simply indicate low values (red) and high values (purple) of whatever variable is used.

The sphere can also be plotted with a Mercator projection, as illustrated in Figure 4. This allows us to see all polarization states simultaneously without the confusion introduced by viewing polarizations on the opposite side of the sphere.

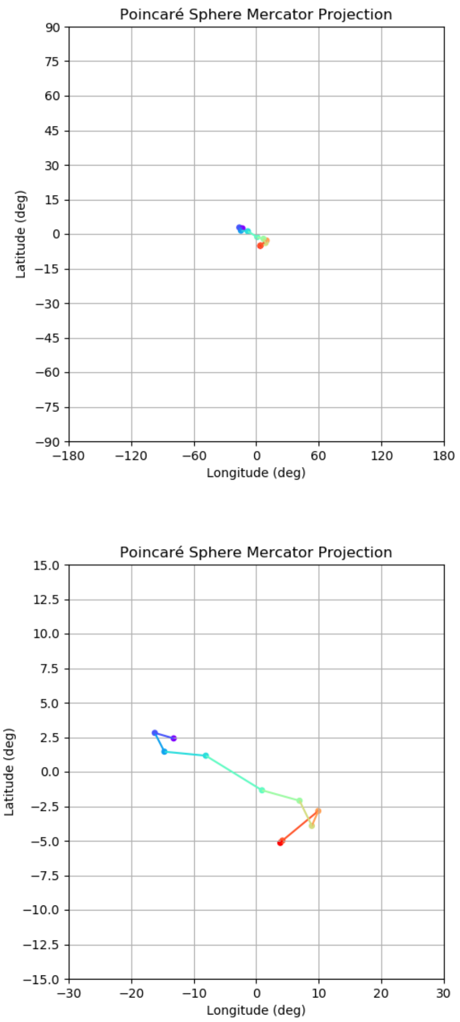

Figure 4. Poincaré sphere plot with Mercator projection showing polarization state vs frequency for a measured antenna (top: full-scale, bottom: zoomed in)

VI. Elliptical Parameters

The polarization state can also be described by characterizing the ellipse that the electric field traces over time at one point in space. A common method of doing this involves defining the following three parameters:

1. Axial ratio: the ratio of major to minor axes of the ellipse
2. Tilt: the angle the major axis makes relative to the x-axis
3. Sense: the direction of rotation of the E-field

These parameters can be represented graphically as shown in Figure 5. The tilt is denoted ψ, the semi-major axis by a, the semi-minor axis by b and the axial ratio would be the ratio of these (i.e. a/b). The sense is the direction of rotation, indicated by arrows pointing clockwise around the ellipse. Clockwise rotation indicates a left-handed sense, and counter-clockwise rotation indicates a right-handed sense.

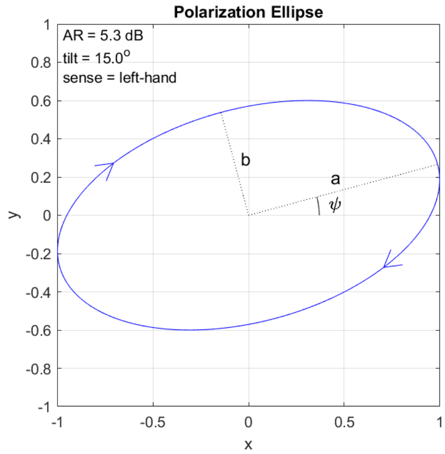

Figure 5. Polarization ellipse associated with $E_1 = 1$ and $E_2 = 0.6e^{j(0.4\pi)}$ with graphical and textual annotations

The axial ratio can be computed as [6]

$$R = 20\log_{10}\left|\frac{|E_{LHCP}| + |E_{RHCP}|}{|E_{LHCP}| - |E_{RHCP}|}\right| \tag{12}$$

Alternatively, it can be computed as [1]

$$R = 20\log_{10}\left|\cot\left|\frac{1}{2}\tan^{-1}\left(\frac{s_3}{\sqrt{s_1^2 + s_2^2}}\right)\right|\right| \tag{13}$$

The tilt can be computed as [1]

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{s_2}{s_1}\right) \tag{14}$$

The sense is computed as [1]

$$\sigma = sign(s_3) \tag{15}$$

Sense is positive when the field rotates clockwise (left-handed) and is negative when the field rotates counter-clockwise (right-handed).

Noting from [1] that the axial ratio is related to the latitude on the sphere and that tilt is related to the longitude on the sphere, we can plot the Mercator projection of the sphere using the commonly known elliptical parameters. More specifically, since Latitude (Lat) and Longitude (Lon) are defined for a point (x,y,z) on the surface of a sphere as

$$Lat = \tan^{-1}\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$

$$Lon = \tan^{-1}\left(\frac{y}{x}\right)$$

We can write the relationship between latitude and longitude and the elliptical parameters as

$$\psi = \frac{1}{2}Lon$$

$$R = 20\log_{10}\left|\cot\left|\frac{1}{2}Lat\right|\right|$$

Informed by these relationships, and knowing that the north pole is LHCP and the south pole is RHCP, the Mercator projection of the sphere is plotted in Figure 6 using tilt and axial ratio as the axes. Note that the upper half of the graph represents positive sense (left-handed) and the lower half negative sense (right-handed).

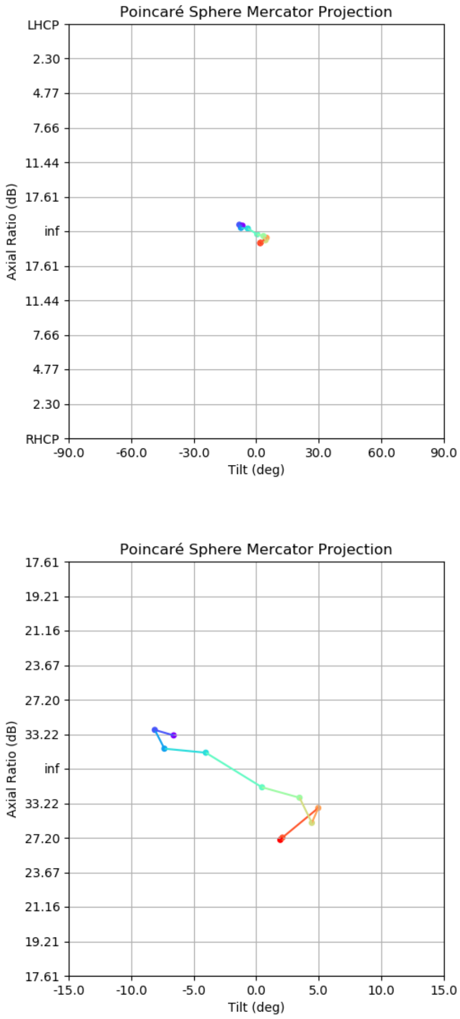

Figure 6. Poincaré sphere plot with Mercator projection using elliptical parameters showing polarization state vs frequency for a measured antenna (top: full-scale, bottom: zoomed in)

VII. Polarization Analyzer Tool

The authors have developed a tool called the Polarization Analyzer for automating the measurement of an antenna and visualizing its polarization state [2]. The use of the tool to acquire real-time polarization data requires the use of an RF source, a dual-polarized probe, and a multi-channel receiver. Alternatively, the tool may be used purely for visualization of the polarization state of previously acquired and/or simulated data. Part of the visualization capability includes the Mercator projection of the Poincaré sphere, as described above. A screenshot of the tool is shown in Figure 7 as an example of what is possible for automating the rendering.

As in Figure 6, the tool shows a set of points connected by multi-colored lines. Each point represents the polarization state at a single frequency, with red being the lowest frequency and purple the highest. Axial ratio is shown on the y-axis and tilt angle on the x-axis. The upper half of the display represents left-hand sense and the bottom half is right-hand sense.

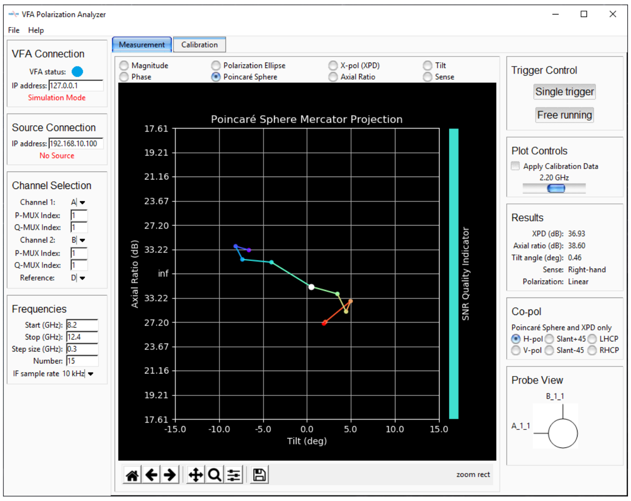

Figure 7. Screenshot of Polarization Analyzer tool

Using this tool, the evolution of the complete polarization state (axial ratio, tilt angle, and sense) with respect to frequency can be readily observed. The display can be updated in real-time, and it can illustrate the differences in behavior before and after calibration of the dual-polarized probe used to measure the polarization of the antenna under test.

VIII. Summary

The Poincaré sphere is a powerful tool for quickly visualizing polarization state across an independent parameter, such as frequency. In spite of its promise, it has often been poorly understood and underutilized. We have reviewed the theory of the sphere and introduced an alternative form for viewing it involving a Mercator projection, a color scheme indicating frequency, and axis labels corresponding to elliptical parameters.

Henri Poincaré, the inventor of the sphere, wrote, “It is by logic that we prove, but by intuition that we discover” [7]. The authors suggest the use of the Poincaré sphere as an aid for building intuition about the polarization states of antennas as a function of different independent variables, such as frequency, aspect angle, design parameter, etc.

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